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2006 J. Phys. A: Math. Gen. 39 14353

(http://iopscience.iop.org/0305-4470/39/46/008)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 14353–14362 doi:10.1088/0305-4470/39/46/008

Integrable properties of a variable-coefficient Korteweg–de Vries model from Bose–Einstein condensates and fluid dynamics

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Received 24 April 2006, in final form 27 September 2006
Published 1 November 2006
Online at stacks.iop.org/JPhysA/39/14353

Abstract

The phenomena of the trapped Bose–Einstein condensates related to matter waves and nonlinear atom optics can be governed by a variable-coefficient Korteweg–de Vries (vc-KdV) model with additional terms contributed from the inhomogeneity in the axial direction and the strong transverse confinement of the condensate, and such a model can also be used to describe the water waves propagating in a channel with an uneven bottom and/or deformed walls. In this paper, with the help of symbolic computation, the bilinear form for the vc-KdV model is obtained and some exact solitonic solutions including the *N*-solitonic solution in explicit form are derived through the extended Hirota method. We also derive the auto-Bäcklund transformation, nonlinear superposition formula, Lax pairs and conservation laws of this model. Finally, the integrability of the variable-coefficient model and the characteristic of the nonlinear superposition formula are discussed.

PACS numbers: 02.30.Ik, 05.45.Yv, 03.75.Lm

1. Introduction

Variable-coefficient Korteweg–de Vries (vc-KdV) models are of current importance in many physical and engineering fields, as seen, e.g., in [1–6] and references therein. Recent study has pointed out that the trapped quasi-one-dimensional Bose–Einstein condensates with repulsive atom–atom interactions and a weak nonlinear excitation can be governed by a vc-KdV model with additional terms contributed from the inhomogeneity in the axial direction and the strong transverse confinement of the condensate [1], i.e.,

$$\mathcal{U}_{\xi\xi\xi} + m_1(X)\mathcal{U}\mathcal{U}_\xi + m_2(X)\mathcal{U}_X + m_3(X)\mathcal{U} + m_4(X)\mathcal{U}_\xi = 0, \tag{1}$$

where X is the scaled space variable in the axial direction, ξ is the multi-scale variable relying on the time and space in the axial direction, $\mathcal{U}(\xi, X)$ represents the modulus of the condensed-state wave function, and the variable coefficients $m_1(X), m_2(X), m_3(X)$ and $m_4(X)$ are all related to the trapping potential in the axial direction and chemical potential. Meanwhile, $m_3(X)$ depends on the inhomogeneity in the axial direction and $m_4(X)$ depends on the transverse confinement of the condensate as depicted in [1]. In addition, such a case is also of value for water waves propagating in a channel with an uneven bottom and/or deformed walls [1, 2, 7].

Describing the nonlinear excitations of a Bose gas of impenetrable bosons with longitudinal confinement [3], a special vc-KdV model without the dissipative term is written as

$$n_{1,X} + f(X)n_1 n_{1,T} + g(X)n_{1,TTT} + l(X)n_1 = 0, \tag{2}$$

where X is the scaled space variable in the axial direction, T is the slow-time variable, $n_1(X, T)$ represents the nonlinear excitation of the gas density, and $l(X)$ is related to the peak density of the gas, while $f(X)$ and $g(X)$ are both functions of the local velocity of excitations and peak density of the gas as explained in [3]. Additionally, this model can also describe the dynamics of a circular rod composed of a general compressible hyperelastic material with variable cross-sections and material density [6], propagation of weakly nonlinear solitonic waves in a varied-depth shallow-water tunnel [8], evolution of internal gravity waves in lakes of changing cross-section [9], etc.

In fact, equation (1) can be transformed to the form

$$u_t + f(t)uu_x + g(t)u_{xxx} + l(t)u + q(t)u_x = 0, \tag{3}$$

through

$$\begin{aligned} &\xi \to x, \qquad X \to t, \qquad \mathcal{U}(\xi, X) \to u(x, t), \qquad \frac{m_1(X)}{m_2(X)} \to f(t), \\ &\frac{1}{m_2(X)} \to g(t), \qquad \frac{m_3(X)}{m_2(X)} \to l(t), \qquad \frac{m_4(X)}{m_2(X)} \to q(t). \end{aligned} \tag{4}$$

Higher-dimensional vc-KdV models can be referred to, e.g. [10].

In [2], with the help of symbolic computation [10, 11], the above models have been transformed into either the standard KdV equation or cylindrical KdV equation under certain constraints. Our primary aims in this paper are educing the bilinear form, analytic N-solitonic solution, auto-Bäcklund transformation, nonlinear superposition formula, Lax pairs and conservation laws of equation (3).

The outline of this paper is as follows. In section 2, we will present the bilinear form and analytic N-solitonic solution for equation (3) via the extended Hirota method. In section 3, we will derive the auto-Bäcklund transformation and nonlinear superposition formula in the bilinear form. In section 4, the Lax pairs and expressions of the conservation laws will be obtained. Section 5 will be our discussions and conclusions.

2. Bilinear form and analytic N-solitonic solution for equation (3)

It is well known that the Hirota method is a powerful tool for dealing with soliton problems for nonlinear evolution equations [12–16], and the variable-coefficient balancing-act method [4, 17] is an effective one to investigate the variable-coefficient nonlinear evolution equations (vc-NLEEs). In our previous works, based on the above two methods, the extended Hirota method has been constructed to transform the vc-NLEEs into variable-coefficient bilinear form and obtain analytic N-solitonic solution of the vc-NLEEs. Hence, equation (3) can be transformed into the variable-coefficient bilinear form

$$\left[D_x D_t + g(t) D_x^4 + q(t) D_x^2\right](F \cdot F) = 0, \tag{5}$$

by the following transformation

$$u(x,t) = C_0\, \mathrm{e}^{-\int l(t)\mathrm{d}t} \frac{\partial^2}{\partial x^2} \ln[F(x,t)], \tag{6}$$

with the constraint

$$f(t) = C_0 g(t)\, \mathrm{e}^{\int l(t)\mathrm{d}t}, \tag{7}$$

where $C_0 \neq 0$ is an arbitrary constant of integration, while constraint (7) is a special case of the condition for equation (3) to pass the Painlevé test[6], and $D_x D_t$, D_x^2 and D_x^4 are all the bilinear operators introduced by [12]

$$D_x^m D_t^n a \cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x,t) b(x',t')\Big|_{x'=x,t'=t}. \tag{9}$$

Based on equation (5) and the method of solving N-soliton solution of the standard KdV equation [18], the dependent variable $F(x,t)$ can be expanded as

$$F(x,t) = 1 + \sum_{i=1}^{n} \epsilon^i F_i(x,t), \tag{10}$$

where ϵ is an arbitrary parameter. Substituting equation (10) into equation (5) and choosing different $F_i (i = 1, 2, \ldots, N)$, we can obtain the analytic N-solitonic solution of equation (3). For instance, the one-solitonic solution can be expressed as

$$\begin{aligned} u(x,t) &= C_0\, \mathrm{e}^{-\int l(t)\,\mathrm{d}t} \left\{ -\frac{4k^2 \exp\left(4kx - 16k^3 \int g(t)\,\mathrm{d}t - 4k \int q(t)\,\mathrm{d}t + 4\delta\right)}{\left[1 + \exp\left(2kx - 8k^3 \int g(t)\,\mathrm{d}t - 2k \int q(t)\,\mathrm{d}t + 2\delta\right)\right]^2} \right. \\ &\quad \left. + \frac{4k^2 \exp\left(2kx - 8k^3 \int g(t)\,\mathrm{d}t - 2k \int q(t)\,\mathrm{d}t + 2\delta\right)}{1 + \exp\left(2kx - 8k^3 \int g(t)\,\mathrm{d}t - 2k \int q(t)\,\mathrm{d}t + 2\delta\right)} \right\} \\ &= C_0 k^2 \mathrm{e}^{-\int l(t)\,\mathrm{d}t}\, \mathrm{Sech}^2\left[kx - 4k^3 \int g(t)\,\mathrm{d}t - k \int q(t)\,\mathrm{d}t + \delta\right], \end{aligned} \tag{11}$$

the two-solitonic solution can be expressed as

$$\begin{aligned} u(x,t) = C_0\, \mathrm{e}^{-\int l(t)\mathrm{d}t} &\left\{ \frac{4{k_1}^2\, \mathrm{e}^{2\theta_1} + 4(k_1 - k_2)^2\, \mathrm{e}^{2(\theta_1+\theta_2)} + 4{k_2}^2\, \mathrm{e}^{2\theta_2}}{1 + \mathrm{e}^{2\theta_1} + \mathrm{e}^{2\theta_2} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\, \mathrm{e}^{2(\theta_1+\theta_2)}} \right. \\ &\left. - \frac{\left[2k_1\, \mathrm{e}^{2\theta_1} + 2k_2\, \mathrm{e}^{2\theta_2} + 2\frac{(k_1-k_2)^2}{k_1+k_2}\, \mathrm{e}^{2(\theta_1+\theta_1)}\right]^2}{\left[1 + \mathrm{e}^{2\theta_1} + \mathrm{e}^{2\theta_2} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\, \mathrm{e}^{2(\theta_1+\theta_2)}\right]^2} \right\}, \end{aligned} \tag{12}$$

[6] One of the authors (GMW) has proved that the condition of equation (3) to pass the Painlevé test is

$$g(t) = f(t)\, \mathrm{e}^{-\int l(t)\mathrm{d}t} \left[\alpha + \beta \int f(t)\, \mathrm{e}^{-\int l(t)\mathrm{d}t} \mathrm{d}t\right]. \tag{8}$$

with $\theta_i = k_i x - 4k_i{}^3 \int g(t)\,\mathrm{d}t - k_i \int q(t)\,\mathrm{d}t + \delta_i (i = 1, 2)$, and the N-solitonic solution ($N = 3, 4, 5, \ldots$) can be written as

$$u(x,t) = C_0\,\mathrm{e}^{-\int l(t)\mathrm{d}t} \frac{\partial^2}{\partial x^2} \ln \left[1 + \sum_{i=1}^{N} F_i(x,t) \right], \tag{13}$$

with $F_i(x,t)$ satisfying

$$F_1 = \sum_{i=1}^{N} \exp\left(2 \left[k_i x - 4k_i{}^3 \int g(t)\,\mathrm{d}t - k_i \int q(t)\,\mathrm{d}t + \delta_i \right] \right), \tag{14}$$

$$F_2 = \sum_{1 \leqslant j < i}^{N} \frac{(k_i - k_j)^2}{(k_i + k_j)^2} \exp\left(2 \left[k_i x - 4k_i{}^3 \int g(t)\,\mathrm{d}t - k_i \int q(t)\,\mathrm{d}t + \delta_i \right.\right.$$
$$\left.\left. + k_j x - 4k_j{}^3 \int g(t)\,\mathrm{d}t - k_j \int q(t)\,\mathrm{d}t + \delta_j \right] \right), \tag{15}$$

$$\cdots\cdots \tag{16}$$

$$F_N = \left[\prod_{1 \leqslant j < i}^{N} \frac{(k_i - k_j)^2}{(k_i + k_j)^2} \right] \exp\left(\left\{ \sum_{i=1}^{N} 2 \left[k_i x - 4k_i{}^3 \int g(t)\,\mathrm{d}t - k_i \int q(t)\,\mathrm{d}t + \delta_i \right] \right\} \right). \tag{17}$$

3. Auto-Bäcklund transformation and nonlinear superposition formula in bilinear form

In this section, we will derive the auto-Bäcklund transformation and nonlinear superposition formula in bilinear form.

Let $F_1(x,t)$ and $F_2(x,t)$ be two distinct solutions of the bilinearized equation, i.e., equation (5), which is

$$\left[D_x D_t + g(t) D_x^4 + q(t) D_x^2\right](F_1 \cdot F_1) = 0, \tag{18}$$

$$\left[D_x D_t + g(t) D_x^4 + q(t) D_x^2\right](F_2 \cdot F_2) = 0, \tag{19}$$

and consider an equation

$$F_1{}^2\left[D_x D_t + g(t) D_x^4 + q(t) D_x^2\right](F_2 \cdot F_2)$$
$$- F_2{}^2\left[D_x D_t + g(t) D_x^4 + q(t) D_x^2\right](F_1 \cdot F_1) = 0. \tag{20}$$

It is obvious that if $F_1(x,t)$ satisfies equation (5), $F_2(x,t)$ also satisfies the same equation. Therefore, equation (20) can be regarded as a relation between the pair of solutions $F_1(x,t)$ and $F_2(x,t)$, namely, the auto-Bäcklund transformation of equation (5).

In order to write equation (20) in the more conventional form with the 'x' part and the 't' part, it is necessary to introduce an arbitrary parameter λ. We can add

$$- 6\lambda D_x[D_x(F_2 \cdot F_1) \cdot (F_1 F_2)] + 6\lambda D_x[(F_1 F_2) \cdot D_x(F_1 \cdot F_2)] \tag{21}$$

to equation (20), yielding

$$F_1{}^2\left[D_x D_t + g(t) D_x^4 + q(t) D_x^2\right](F_2 \cdot F_2) - 6\lambda D_x[D_x(F_2 \cdot F_1) \cdot (F_1 F_2)]$$
$$- F_2{}^2\left[D_x D_t + g(t) D_x^4 + q(t) D_x^2\right](F_1 \cdot F_1) + 6\lambda D_x[(F_1 F_2) \cdot D_x(F_1 \cdot F_2)] = 0. \tag{22}$$

Via some properties of the bilinear operators [12, 14, 15], equation (22) can be written as

$$
\begin{aligned}
&2D_x[D_t(F_2\cdot F_1)\cdot(F_1F_2)]+2g(t)D_x\big[D_x^3(F_2\cdot F_1)\cdot(F_1F_2)\big]\\
&\quad+2q(t)D_x[D_x(F_2\cdot F_1)\cdot(F_1F_2)]+6g(t)D_x\big[D_x^2(F_2\cdot F_1)\cdot D_x(F_1\cdot F_2)\big]\\
&\quad-6\lambda D_x[D_x(F_2\cdot F_1)\cdot(F_1F_2)]+6\lambda D_x[(F_1F_2)\cdot D_x(F_1\cdot F_2)]=0.
\end{aligned}
\tag{23}
$$

Combining the first, second, third and fifth terms on the left-hand side of equation (23) and other two terms respectively, we have

$$
\begin{aligned}
&2D_x\big\{\big[D_t(F_2\cdot F_1)+g(t)D_x^3(F_2\cdot F_1)+q(t)D_x(F_2\cdot F_1)-3\lambda D_x(F_2\cdot F_1)\big]\cdot(F_1F_2)\big\}\\
&\quad+D_x\big\{\big[6g(t)D_x^2(F_2\cdot F_1)+6\lambda(F_2F_1)\big]\cdot D_x(F_1\cdot F_2)\big\}=0.
\end{aligned}
\tag{24}
$$

Obviously, equation (24) is satisfied provided that the following equations hold:

$$
\big[D_t+g(t)D_x^3+q(t)D_x-3\lambda D_x\big](F_2\cdot F_1)=0,
\tag{25}
$$

$$
g(t)D_x^2(F_2\cdot F_1)+\lambda(F_2F_1)=0,
\tag{26}
$$

which constitute the auto-Bäcklund transformation of equation (3) in bilinear form, where equations (25) and (26) equate with the 't' part and 'x' part, respectively.

In addition, if we add

$$
-6g(t)\lambda D_x[D_x(F_2\cdot F_1)\cdot(F_1F_2)]+6g(t)\lambda D_x[(F_1F_2)\cdot D_x(F_1\cdot F_2)]
\tag{27}
$$

to equation (20), then we can derive another form of the auto-Bäcklund transformation

$$
\big[D_t+g(t)D_x^3+q(t)D_x-3g(t)\lambda D_x\big](F_2\cdot F_1)=0,
\tag{28}
$$

$$
\big[D_x^2+\lambda\big](F_2\cdot F_1)=0.
\tag{29}
$$

To illustrate, taking $F_1=1$ as a vacuum solution of equation (5) and substituting it into the auto-Bäcklund transformation, i.e., equations (28) and (29) (or equations (25) and (26)), yield

$$
F_2=\mathrm{e}^{-\theta}(1+\mathrm{e}^{2\theta})\qquad\text{with}\qquad \theta=kx-4k^3\int g(t)\,\mathrm{d}t-k\int q(t)\,\mathrm{d}t+\delta.
\tag{30}
$$

Substitution of equation (30) into equation (6) gives rise to a solution of equation (3) as seen

$$
u(x,t)=C_0k^2\,\mathrm{e}^{-\int l(t)\mathrm{d}t}\,\mathrm{Sech}^2\left[kx-4k^3\int g(t)\,\mathrm{d}t-k\int q(t)\,\mathrm{d}t+\delta\right],
\tag{31}
$$

which is the same as expression (11), i.e. the one-solitonic solution of equation (3).

Next, we will present the nonlinear superposition formula.

It should be noted that equation (29), the 'x' part of the auto-Bäcklund transformation, is identical to the one of the standard KdV equation [19], hereby, we can obtain the nonlinear superposition formula of equation (3) similar to that of the standard KdV equation [19]. Let $F_0(x,t)$ be a solution of equation (5), we can obtain two solutions $F_1(x,t)$ and $F_2(x,t)$ via the auto-Bäcklund transformation, i.e. equations (28) and (29), with two arbitrary parameters λ_1 and λ_2 ($\lambda_1\neq\lambda_2$). Then the nonlinear superposition formula is written as

$$
F_{12}=CF_0^{-1}D_x(F_1\cdot F_2),
\tag{32}
$$

where $C\neq 0$ is an arbitrary constant.

4. Conservation laws and Lax pairs

In the study of a system of partial differential equations (PDEs), the concept of a conservation law, which is a mathematical formulation of the familiar physical laws such as conservation of energy and conservation of momentum, plays an important role in the analysis of the basic properties of the solutions [20]. Hence, in this section, we will study the existence of an infinite number of conservation laws and Lax pairs of equation (3).

Equation (3) can be equivalently written as

$$u'_{t'} - 6u'u'_{x'} + u'_{x'x'x'} = 0 \tag{33}$$

from transformations $x' \longleftrightarrow x - \int q(t)\,\mathrm{d}t$, $t' \longleftrightarrow \int g(t)\,\mathrm{d}t$, $u'(x', t') \longleftrightarrow \mathrm{e}^{\int -l(t)\,\mathrm{d}t} u(x, t)$ and $C_0 = -6$ without loss of generality in equation (7).

As we know, the existence of the infinite conservation laws of equation (33) has been proved in [21, 22]. It should be noted that equation (3), equivalent to equation (33) via the above transformations, also possesses the infinite conservation laws. In illustration, the first three conservation laws of equation (3), i.e. conservation of mass, conservation of momentum and conservation of energy, are presented in detail. The corresponding conserved densities are given by

$$T_1 = \mathrm{e}^{\int l(t)\mathrm{d}t}\, u, \tag{34}$$

$$T_2 = \tfrac{1}{2}\,\mathrm{e}^{2\int l(t)\mathrm{d}t}\, u^2, \tag{35}$$

$$T_3 = \mathrm{e}^{3\int l(t)\mathrm{d}t}\, u^3 + \tfrac{1}{2}\,\mathrm{e}^{2\int l(t)\mathrm{d}t}\, {u_x}^2, \tag{36}$$

and the fluxes can be written as

$$X_1 = \mathrm{e}^{\int l(t)\mathrm{d}t}\left[q(t)u + \tfrac{1}{2} f(t)u^2 + g(t)u_{xx}\right], \tag{37}$$

$$X_2 = \mathrm{e}^{2\int l(t)\mathrm{d}t}\left[\tfrac{1}{2}q(t)u^2 + \tfrac{1}{3} f(t)u^3 - \tfrac{1}{2}g(t){u_x}^2 + g(t)uu_{xx}\right], \tag{38}$$

$$\begin{aligned} X_3 = {} & \mathrm{e}^{3\int l(t)\mathrm{d}t} q(t)u^3 + 3\,\mathrm{e}^{3\int l(t)\mathrm{d}t} g(t)u^2u_{xx} + \tfrac{3}{4}\,\mathrm{e}^{3\int l(t)\mathrm{d}t} f(t)u^4 \\ & + \mathrm{e}^{2\int l(t)\mathrm{d}t}\left[\tfrac{1}{2}q(t){u_x}^2 + f(t)u{u_x}^2 - \tfrac{1}{2}g(t){u_{xx}}^2 + g(t)u_xu_{xxx}\right]. \end{aligned} \tag{39}$$

It is easy to prove that $\frac{\partial T_i}{\partial t} + \frac{\partial X_i}{\partial x} = 0\ (i = 1, 2, 3)$.

Next, we will give the Lax pairs of equation (3).

Without loss of generality, taking $C_0 = -6$ in equation (7), the Lax pair can be obtained through the AKNS system [23–26],

$$L = -D^2 + \mathrm{e}^{\int l(t)\mathrm{d}t} u, \tag{40}$$

$$M = -4g(t)D^3 - f(t)uD - q(t)D + 3g(t)\,\mathrm{e}^{\int l(t)\mathrm{d}t} u_x, \tag{41}$$

where $D = \frac{\partial}{\partial x}$. It is easy to see that $L_t - [M, L] = 0$ as follows:

$$L_t = l(t)\,\mathrm{e}^{\int l(t)\mathrm{d}t} u + \mathrm{e}^{\int l(t)\mathrm{d}t} u_t, \tag{42}$$

$$\begin{aligned} [M, L] &= ML - LM \\ &= -g(t)\,\mathrm{e}^{\int l(t)\mathrm{d}t} u_{xxx} - f(t)\,\mathrm{e}^{\int l(t)\mathrm{d}t} uu_x - q(t)\,\mathrm{e}^{\int l(t)\mathrm{d}t} u_x, \end{aligned} \tag{43}$$

$$L_t - [M, L] = \mathrm{e}^{\int l(t)\mathrm{d}t}\left[u_t + f(t)uu_x + g(t)u_{xxx} + l(t)u + q(t)u_x\right] = 0. \tag{44}$$

In addition, other Lax representations when $C_0 = -6$ can also be found from the AKNS system [23–26], for instance,

$$L = -D^2 + \mathrm{e}^{\int l(t)\mathrm{d}t} u, \tag{45}$$

$$M = -g(t)D^3 + \tfrac{1}{2} f(t)\,\mathrm{e}^{-\int l(t)\mathrm{d}t} D^3 - f(t)uD - q(t)D - \tfrac{1}{2} f(t)u_x. \tag{46}$$

It should be noted that other formats for Lax pairs which are different from the above expressions also exist if $C_0 \neq -6$.

5. Discussions and conclusions

The vc-KdV models with perturbed and dissipative terms have been widely used in physical and engineering sciences. For instance, those models can describe the trapped quasi-one-dimensional Bose–Einstein condensates with repulsive atom–atom interactions and a weak nonlinear excitation [1], water waves in a channel with an uneven bottom and/or deformed walls [1, 7], nonlinear excitations of a Bose gas of impenetrable bosons with longitudinal confinement [3], dynamics of a circular rod composed of a general compressible hyperelastic material with variable cross sections and material density [6], propagation of weakly nonlinear solitonic waves in a varied-depth shallow-water tunnel [8], evolution of internal gravity waves in lakes of changing cross section [9], etc. In the above sections, by using Mathematica, many properties of the aforementioned models have been presented such as the bilinear form, N-solitonic solution in explicit form, auto-Bäcklund transformation, nonlinear superposition formula, Lax pairs and conservation laws.

The discussions and conclusions on the above results are as follows.

(1) Since we have derived the bilinear form, N-solitonic solution in explicit form, auto-Bäcklund transformation, nonlinear superposition formula, Lax pairs and conservation laws for equation (3) under constraint (7), it can be concluded that equation (3) is integrable when the coefficient functions of nonlinear, dispersive and perturbed terms satisfy constraint (7). One of the authors GMW has obtained the more general condition for equation (3) to pass the Painlevé test (see footnote 6). But whether equation (3) under the more general condition possesses the integrable properties as we have presented above when constraint (7) holds need to be further investigated. Special attention should be paid to the fact that constraint (7) is only on the coefficient functions $f(t)$, $g(t)$ and $l(t)$, but has nothing to do with $q(t)$, i.e. the coefficient of the dissipative term.

(2) From the expression of 'x' part of the auto-Bäcklund transformation, i.e. equation (29), it should be noted that equation (29) is identical to the one of the standard KdV equation. Meanwhile, the nonlinear superposition formula in bilinear form is also identical to the standard one. We may suppose that the nonlinear superposition formula for other generalized vc-KdV equations with perturbed and/or dissipative terms and/or external force is also identical to equation (32).

(3) Physically speaking, the analytic solutions of the vc-KdV models, especially the solitonic solutions, might help us to better understand the physical mechanisms such as the Bose–Einstein condensates for equation (1) and the Bose gas density for equation (2). Although the coefficient functions and additional terms make it hard to study the vc-KdV equation, via symbolic computation and the extended Hirota method, the N-solitonic solution in explicit form can be gained. It is found that the amplitude of the solitary wave will decrease or increase because of the accumulated effect of perturbation (via $l(t)u$) over a time period. We can also find that the coefficient functions of the nonlinear,

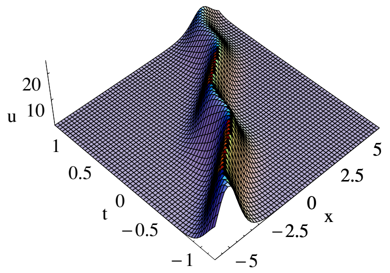

Figure 1. Solution surface $u(x, t)$ describing one solitary wave via expression (11) with $k = 1.0$, $C_0 = 2$, $g(t) = 1 + 0.6\sin(6t)$, $q(t) = 0.5 + 0.4\sin(6t)$, $l(t) = 0.5 + 0.4\sin(6t)$.

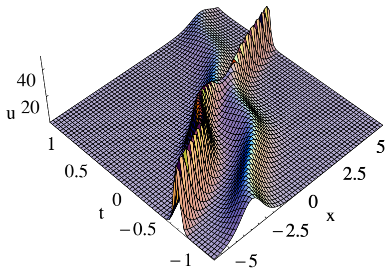

Figure 2. Solution surface $u(x, t)$ describing the nonlinear interaction of two solitary waves via expression (12) with $k_1 = 1.0$, $k_2 = 1.5$ and the others are the same as figure 1.

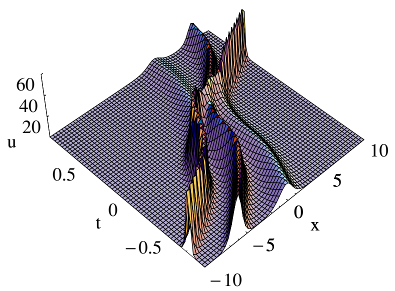

Figure 3. Solution surface $u(x, t)$ describing the nonlinear interaction of three solitary waves via expression (13) with $k_1 = 0.9$, $k_2 = 1.4$, $k_3 = 1.9$ and the others are the same as figure 1.

dispersive and dissipative terms, i.e. $f(t)$, $g(t)$ and $q(t)$, are the primary influential factors for the variation of the solitary wave shape. The above phenomena can be seen from figures 1, 2 and 3. It is noted that those solitary waves are beyond the travelling waves.

(4) Although the coefficient functions have influences on the vc-KdV system under investigation, the N-solitonic solution for equation (3) under constraint (7) admits many properties which are similar to those of the standard KdV equation. For instance, the collision between two solitary waves or among three solitary waves (even N solitary waves) is compatible with elastic collision, that is to say, after the collision the solitary waves retain their original shapes with the moment of the collision being a phase shift,

as the interaction between two solitary waves investigated in [28, 29]. Figures 2 and 3, respectively, show the two and three solitary waves nonlinear interactions characteristic of the phase shifts. It is obvious in figure 3 that the tallest solitary wave catches the shorter and the shorter catches the shortest, and then, the waves exchange roles at the moment of the collision, so that the tallest amplitude becomes shorter and the shortest amplitude becomes taller before they separate.

(5) To sum up, the vc-KdV models from Bose–Einstein condensates and fluid dynamics possess many integrable properties under constraint (7) which can be regarded as a sufficient condition for those models to be integrable.

Acknowledgments

We express our sincere thanks to the referees for their valuable suggestions. This work has been supported by the Key Project of Chinese Ministry of Education (No 106033), and by the National Natural Science Foundation of China under grants no 60372095 and no 10272017. CYZ thanks the Green Path Program of Air Force of the Chinese People's Liberation Army. YTG would like to acknowledge the Cheung Kong Scholars Programme of the Ministry of Education of China and Li Ka Shing Foundation of Hong Kong.

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